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Fermions and emergent noncommutative gravity

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Abstract: Fermions coupled to Yang-Mills matrix models are studied from the point of view of emergent gravity. We show that the simple matrix model action provides an appropriate coupling for fermions to gravity, albeit with a non-standard spin connection. Integrating out the fermions in a nontrivial geometrical background induces indeed the Einstein-Hilbert action and a dilaton-like term, at least for on-shell geometries. This explains and precisely reproduces the UV/IR mixing for fermions in noncommutative gauge theory, extending recent results for scalar fields. It also explains why some UV/IR mixing remains even in supersymmetric models, except in the $N = 4$ case.

Keywords: M(atrix) Theories, Gauge-gravity correspondence, Non-Commutative Geometry, Models of Quantum Gravity.

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Contents

1. Introduction

Recently it was understood that gravity can arise effectively from non-commutative gauge theory [1 – 3]. This mechanism is realized in certain matrix models of Yang-Mills type, which are known to describe gauge theory on non-commutative (NC) spaces. These models can be interpreted more naturally as describing dynamical NC spaces, suggesting an intrinsic realization of some sort of gravity theory. The mechanism of this "emergent gravity" is quite simple and intrinsically non-commutative [1]: On a general background corresponding to a generic NC space, all fields in the model couple universally (up to possibly density factors) to an effective metric or frame associated with the background. The Einstein-Hilbert action is induced upon quantization. Such a generic NC background can be equivalently interpreted in terms of a $\mathrm{U}(1)$ gauge field on a fixed Moyal-Weyl space $\mathbb{R}^4_\theta$. This then provides an understanding of UV/IR mixing in the one-loop effective action of NC gauge theory, in terms of an induced gravity action.

In the present paper, we include fermions in these matrix models, and study their coupling to the emergent gravity. In particular, we extend the explanation of UV/IR mixing in terms of induced gravity [4] to the case of fermions. The matrix-model framework strongly suggests a simple action for fermions, as realized e.g. in the string-theoretical IKKT model. We study this fermionic action in detail on a generic NC background, following the geometrical point of view of [1]. This leads to an effective semi-classical action for a fermion on a background with metric $G_{ab}(y)$. The action is similar to the standard action for fermions on a curved background, except that the spin-connection vanishes in a preferred frame, which arises in emergent NC gravity as defined through the matrix model.

The main result of this paper is that in spite of this different spin connection, the simple fermionic action under consideration is nevertheless reasonable on general backgrounds, and should be physically viable. In the point particle limit, the fermions will follow the same trajectories as for the conventional coupling to gravity, albeit with a different rotation of the spin along the trajectory. Furthermore, we determine the effective gravitational action obtained by integrating out the fermions in our framework. This leads indeed to the expected Einstein-Hilbert action for the effective metric, with an extra term for a density factor, and another term which vanishes on-shell. We conclude that the fermionic action considered here is the "correct" one for emergent gravity, suitable for a physically realistic theory of gravity.

For a consistent quantization of emergent gravity, a supersymmetric extension of the model appears to be necessary. We point out that the induced gravitational action will be finite if and only if the model enjoys $N = 4$ supersymmetry at the Planck scale. This explains in particular the persistence of UV/IR mixing in NC gauge theory in models with less supersymmetry. Such a supersymmetric extension is realized by the IKKT model, which is known to admit noncommutative backgrounds; see e.g. [5 − 8] and references therein for related work including (indirect) evidence for gravity on such backgrounds.

This paper is organized as follows. In section 2, we review the basic aspects of emergent gravity, and write down the actions under consideration. In section 3, we review the mechanism of induced gravity, and set up the form of the appropriate Seeley-de Witt coefficients corresponding to the fermionic action. They are the ingredients which provide the induced effective gravity action, and they are modified in comparison with the commutative case. These coefficients are computed in section 4, which is the core of this paper. We also provide an exact expression for the Ricci scalar in terms of the effective metric of emergent gravity; this should be useful for other considerations as well. In section 5, we consider the same problem from the point of view of NC gauge theory, and rewrite the obtained gravitational action in terms of U(1) gauge fields on Moyal-Weyl space. The one-loop effective action for this NC gauge theory is carefully computed in the appropriate IR limit, which provides a precise matching with the gravitational action obtained before. We conclude with a short discussion of supersymmetry and an outlook.

2. Matrix models and effective geometry

As a starting point of emergent NC gravity, we consider the semi-classical geometry arising from the matrix model with action

$$S_{\rm YM} = -Tr[Y^a, Y^b][Y^{a'}, Y^{b'}] g_{aa'} g_{bb'} \,. \tag{2.1}$$

Here $Y^a,\ a = 1,2,3,4$ are hermitian matrices or operators acting on some Hilbert space $\mathcal{H}$, which constitute the dynamical objects of the model. The model also contains a constant background metric, which is

$$g_{aa'} = \delta_{aa'} \quad \text{or} \quad g_{aa'} = \eta_{aa'} \tag{2.2}$$

in the Euclidean resp. Minkowski case. The commutator of 2 matrices is denoted as

$$[Y^a, Y^b] = i\theta^{ab}\,, \tag{2.3}$$

defining $\theta^{ab} \in L(\mathcal{H})$ as an antihermitian operator-valued matrix. We focus here on configurations where the Y^a (which need not be solutions of the equation of motion) can be interpreted as quantization of coordinate functions y^a on a Poisson manifold $(\mathcal{M}, \theta^{ab}(y))$ with general Poisson structure $\theta^{ab}(y)$. This defines the geometrical background under consideration, and conversely any Poisson manifold provides (locally) after quantization a possible background Y^a. More formally, this means that there is an isomorphism of vector spaces

$$\begin{array}{rcl} \mathcal{C}(\mathcal{M}) & \to & \mathcal{A} \subset L(\mathcal{H}) \\ f(y) & \mapsto & \hat{f}(Y) \\ i\{f,g\} & \mapsto & [\hat{f},\hat{g}] + O(\theta^2) \end{array} \tag{2.4}$$

where $\mathcal{C}(\mathcal{M})$ denotes the space of functions on $\mathcal{M}$, and $\mathcal{A}$ is the algebra generated by Y^a, interpreted as quantized algebra of functions. Thus θ^{ab} reduces in a semi-classical limit to a classical Poisson structure $\theta^{ab}(y)$. Observe that the Y^a correspond to preferred coordinate functions y^a on $\mathcal{M}$, and $g_{ab} = \delta_{ab}$ resp. $g_{ab} = \eta_{ab}$ defines a flat "background" metric, which is constant in the preferred coordinates y^a. Indices throughout this work will always refer to these preferred coordinates,[1] and only some (final) formulas will be manifestly covariant.

To simplify things, we restrict ourselves to the purely geometrical or "irreducible" case in this paper, i.e. we assume that $\mathcal{A}$ is in some sense dense in $L(\mathcal{H})$. This means that any matrix in $L(\mathcal{H})$ can be considered as a function of Y^a resp. y^a. From the gauge theory point of view in section 5, it means that we restrict ourselves to the U(1) case; this is most interesting here since the UV/IR mixing is restricted to the trace-U(1) sector. For the nonabelian case see [1].

Besides the preferred coordinates y^a, the noncommutative background provides in the semi-classical limit a preferred frame

$$e^a = -i[Y^a, .] = \theta^{ab}\partial_b \tag{2.5}$$

given in terms of the antisymmetric Poisson tensor, which we assume to be non-degenerate. This formula is only valid for the preferred coordinates y^a, and does not admit local Lorentz transformations. The effective metric arising from the matrix model turns out to be [1]

$$G^{ab}(y) = \theta^{ac}(y)\theta^{bd}(y)\, g_{cd}\,, \tag{2.6}$$

[1] this is the reason why many of the formulas in this paper are written in a non-covariant way. A covariant formulation is possible but will not be given here.

which has the correct tensor structure and is associated to the above frame. $G^{ab}(y)$ is indeed the effective gravitational metric in the matrix model (up to a rescaling discussed below), because it enters the kinetic terms for matter fields through $[Y^a,\Psi] \sim i\theta^{ab}(y)\frac{\partial}{\partial y^b}\Psi$; this will be seen explicitly below. Note that $G^{ab}(y)$ is not flat in general. Hence the background $\mathcal{M}$ naturally acquires a metric structure $(\mathcal{M},\theta^{ab}(y),G^{ab}(y))$, which is determined by the Poisson structure and the constant background metric g_{ab}. While this is rather obvious for scalar fields, it turns out that essentially the same G^{ab} also couples to nonabelian gauge fields [1], and to fermions as we will show here.

An infinitesimal version of the metric (2.6) was observed already in [2]. The frame (2.5) was also pointed out in [3], as well as a metric of type (2.6) for the self-dual case. There is also some overlap with the ideas in [9]. For further related work see e.g. [10–12].

Scalars. We first review the case of scalar fields i.e. hermitian matrices Φ coupled to the matrix model (2.1). The only possibility to write down kinetic terms for matter fields is through commutators $[Y^a,\Phi] \sim i\theta^{ab}(y)\frac{\partial}{\partial y^b}\Phi$, and one is lead to the action

$$\begin{aligned} S[\Phi] &= -(2\pi)^2\,\mathrm{Tr}\, g_{aa'}[Y^a,\Phi][Y^{a'},\Phi] \\ &\sim \int d^4y\,\rho(y)\,G^{ab}(y)\,\frac{\partial}{\partial y^a}\Phi(y)\frac{\partial}{\partial y^b}\Phi(y). \end{aligned} \tag{2.7}$$

Here and throughout this paper, $\sim$ indicates the leading contribution in a semi-classical expansion in powers of θ^{ab}. This involves the symplectic measure

$$\rho(y) = (\det\theta^{ab}(y))^{-1/2} = |G_{ab}(y)|^{1/4} \equiv e^{-\sigma} \quad (\equiv \Lambda^4_{\mathrm{NC}}(y)) \tag{2.8}$$

on $(\mathcal{M},\theta^{ab}(y))$, which can be naturally interpreted as non-commutative scale Λ^4_{NC}; here $|G_{ab}(y)| \equiv |\det G_{ab}(y)|$. After appropriate rescaling of $G^{ab}(y)$, this can be rewritten in covariant form

$$S[\Phi] = \int d^4y\,\tilde{G}^{ab}(y)\partial_{y^a}\Phi\partial_{y^b}\Phi \tag{2.9}$$

with the effective metric

$$\begin{aligned} \tilde{G}^{ab} &= |G_{ab}|^{1/4}\,G^{ab} = \rho(y)\,G^{ab} \\ |\tilde{G}^{ab}| &= 1 \end{aligned} \tag{2.10}$$

which is unimodular in the preferred y^a coordinates. Recall that these are characterized by the background metric being constant, $g_{ab} = \delta_{ab}$ resp. $g_{ab} = \eta_{ab}$.

Fermions. Then the most obvious (perhaps the only reasonable) action for a spinor which can be written down in the matrix model framework[2] is

$$S = (2\pi)^2\,\mathrm{Tr}\overline{\Psi}\gamma_a[Y^a,\Psi] \;\sim\; \int d^4y\,\rho(y)\,\overline{\Psi}i\gamma_a\theta^{ab}(y)\partial_b\Psi \tag{2.11}$$

[2]In particular, fermions should also be in the adjoint, otherwise they cannot acquire a kinetic term. This does not rule out its applicability in particle physics, see e.g. [14].

ignoring possible nonabelian gauge fields here to simplify the notation. This is written for the case of Minkowski signature, the Euclidean version involves the obvious replacement $\bar{\Psi} \to \Psi^\dagger$. This defines the (matrix) Dirac operator

$$\not{D}\,\Psi = \gamma_a[Y^a, \Psi] \;\;\sim\;\; i\gamma_a\theta^{ab}(y)\partial_b\Psi. \tag{2.12}$$

We can compare this with the standard covariant derivative for spinors (see e.g. [13])

$$\not{D}_{\rm comm}\,\Psi = i\gamma^a\, e_a^\mu\left(\partial_\mu + \Sigma_{bc}\,\omega_\mu^{bc}\right)\Psi \tag{2.13}$$

where

$$\omega_\mu^{ab} = i\frac{1}{2}\,e^{a\nu}\left(\nabla_\mu e_\nu^b\right) \tag{2.14}$$

is the spin connection, and

$$\Sigma_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]\,. \tag{2.15}$$

Comparing (2.12) with (2.13), we observe again that in the geometry defined by (2.6),

$$e_b^\mu(y) := \theta^{\mu c}(y)g_{cb} \tag{2.16}$$

plays the role of a preferred vielbein. However this must be used with great care, because the distinction between the coordinate index μ and the Lorentz index a is lost in the special "gauge" inherent in (2.16).

One notices immediately that the Dirac operator (2.12) has the standard kinetic term corresponding to the metric G^{ab} (cf. (3.5)), but a non-standard spin connection which vanishes in the preferred frame (2.16). One of the main messages of this work is that in spite of this strange feature, (2.11) defines a good action for a fermion propagating in the geometry defined by $\tilde{G}_{ab}$. Recall that the spin connection determines how the spinors are rotated under parallel transport along a trajectory. However, the spin-connection ω_μ^{ab} can always be eliminated (via parallel-transport resp. a suitable gauge choice) along an open trajectory. Then the conventional kinetic term (2.13) boils down to (2.11). Therefore in the point-particle limit, the trajectory of a fermion with action (2.11) will follow properly the geodesics of the metric[3] $\tilde{G}_{ab}$, albeit with a different rotation of the spin. Furthermore, we will show in detail that the induced gravitational action obtained by integrating out the fermion in (2.11) indeed induces the expected Einstein-Hilbert term $\int d^4y\, R[\tilde{G}]\,\Lambda^2$ at least for "on-shell geometries", albeit with an unusual numerical coefficient and an extra term depending on σ. All this shows that (2.11) defines a reasonable action for fermions in the background defined by $\tilde{G}_{ab}$.

In particular, the transport along a closed curve determines the holonomy in a gravitational field, and the vanishing spin connection in the above action strongly suggests that holonomies here will be different than in General Relativity. More generally, the gravitational spin rotation for a free-falling fermion might provide a nice signature for or against the emergent gravity framework.

[3]for massless particles, the geodesics of $\tilde{G}_{ab}$ coincide with those of G_{ab}. Masses should be generated spontaneously, which is not considered here.

Equations of motion. So far we considered arbitrary background configurations Y^a as long as they admit a geometric interpretation. The equations of motion derived from the action (2.1) are

$$[Y^a, [Y^{a'}, Y^b]]\, g_{aa'} = 0\,, \tag{2.17}$$

which in the semi-classical limit amount to

$$\theta^{ma}\partial_m\theta^{nb}g_{ab} = 0 \qquad \text{or} \qquad \tilde{G}^{ac}\partial_c\theta^{-1}_{cd} = 0\,. \tag{2.18}$$

They select on-shell geometries among all possible backgrounds, such as the Moyal-Weyl quantum plane (5.2). In the present geometric form they amount to Ricci-flat spaces [2, 1] at least in the linearized case. However since we are interested in the quantization here, we will need general off-shell configurations below. The equations (2.18) are only valid in the preferred y^a coordinates as discussed above, and the same applies to most computations below. They can be cast in covariant form, which will not be done here.

3. Quantization and induced gravity

We are interested in the quantization of our matrix model coupled to fermions. In principle, the quantization is defined in terms of a ("path") integral over all matrices Y^a and Ψ. In four dimensions, we can only perform perturbative computations for the "gauge sector" Y^a, while the fermions can be integrated out formally in terms of a determinant. Let us focus here on the effective action Γ_Ψ obtained by integrating out the fermionic fields,

$$e^{-\Gamma_\Psi} = \int d\Psi\, d\bar{\Psi}\; e^{-S[\Psi]}, \tag{3.1}$$

which for non-interacting fermions is given by

$$\Gamma_\Psi = -\frac{1}{2}\mathrm{Tr}\log \not{D}^2. \tag{3.2}$$

Later, we will consider an alternative interpretation as Laplacian of the fermionic fields on $\mathbb{R}^4_\theta$ coupled to an adjoint U(1) gauge field. In Feynman diagram language, (3.2) will then amount to the sum of all one-loop diagrams with arbitrary numbers of external A-lines.

In [4], the analogous computation of the effective action Γ_Φ for scalar fields was carried out in two different ways. On the one hand Γ_Φ was evaluated from the geometric point of view via induced Einstein-Hilbert action. On the other hand the action (2.7) was regarded as NC gauge theory by means of covariant coordinates, where the effective action for the NC gauge fields was determined by integrating out the scalar fields. The obtained effective actions were shown to agree in the IR regime as expected. As a consequence UV/IR mixing of NC gauge theory can be interpreted as an effect of gravity.

In this paper, we generalize the results of [4] to the case of fermions. The effective action Γ_Ψ is evaluated in two different ways: We first compute Γ_Ψ as induced gravity action, using its semi-classical geometrical form. This is then compared with the one-loop effective action of NC $\mathfrak{u}(1)$ gauge theory. As expected, we find complete agreement in a suitable IR regime.

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Square of the Dirac operator and induced action. Starting from the action

$$S = (2\pi)^2 \,\mathrm{Tr}\Psi^\dagger\gamma_a[Y^a,\Psi] \tag{3.3}$$

we want to study

$$\begin{aligned} e^{-\Gamma_\Psi} &= \int d\Psi d\overline{\Psi}\, e^{-(2\pi)^2\, Tr\Psi^\dagger\gamma_a[Y^a,\Psi]} \;\cong\; \det(\gamma_a[Y^a,.]) \\ &= \exp\left(\ln\det(\not{D})\right) = \exp\left(\frac{1}{2}\log\det(\not{D}^2)\right) \\ &= \exp\left(\frac{1}{2}\mathrm{Tr}\log(\not{D}^2)\right) \end{aligned} \tag{3.4}$$

at one loop, considering the Euclidean case for the sake of rigor. The square of the Dirac operator takes the following form

$$\begin{aligned} \not{D}^2\Psi &= \gamma_a\gamma_b[Y^a,[Y^b,\Psi]] \\ &= -\gamma_a\gamma_b\theta^{ac}\partial_c(\theta^{bd}\partial_d\Psi) \\ &= -G^{cd}\partial_c\partial_d\Psi - a^d\partial_d\Psi, \end{aligned} \tag{3.5}$$

with

$$a^d = \gamma_a\gamma_b\theta^{ma}\partial_m\theta^{db} = -2i\,\Sigma_{ab}\theta^{ac}\partial_c\theta^{bd} + g_{ab}\theta^{ac}\partial_c\theta^{bd}. \tag{3.6}$$

See appendix C for a comparison of this term with the commutative case. $\not{D}^2$ defines the quadratic form

$$\begin{aligned} S_{\text{square}} &= (2\pi)^2\,\mathrm{Tr}\Psi^\dagger\,\not{D}^2\Psi \sim \int d^4y\,\rho(y)\Psi^\dagger\,\not{D}^2\Psi \\ &= \int d^4y\,|G_{ab}|^{1/4}\Psi^\dagger\,\not{D}^2\Psi, \end{aligned} \tag{3.7}$$

which is very similar to the scalar action. In terms of the metric $\tilde{G}_{ab}$ (2.10) with $|\tilde{G}_{ab}| = 1$, S_{square} can be written in covariant form

$$S_{\text{square}} = \int d^4y\sqrt{|\tilde{G}|}\;\overline{\Psi}\,\widetilde{\not{D}^2}\Psi, \tag{3.8}$$

in terms of the rescaled squared Dirac operator

$$\widetilde{\not{D}^2}\Psi = -\left(\tilde{G}^{cd}\partial_c\partial_d\Psi + e^{-\sigma}\,a^d\partial_d\Psi\right). \tag{3.9}$$

We now compute the effective action using

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}\left(\log\widetilde{\not{D}^2} - \log\widetilde{\not{D}^2_0}\right) &= -\frac{1}{2}\mathrm{Tr}\int_0^\infty d\alpha\frac{1}{\alpha}\left(e^{-\alpha\widetilde{\not{D}^2}} - e^{-\alpha\widetilde{\not{D}^2_0}}\right) \\ &\equiv\; -\frac{1}{2}\mathrm{Tr}\int_0^\infty\frac{d\alpha}{\alpha}\left(e^{-\alpha\widetilde{\not{D}^2}} - e^{-\alpha\widetilde{\not{D}^2_0}}\right)e^{-\frac{1}{2\alpha\tilde{\Lambda}^2}}, \end{aligned} \tag{3.10}$$

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where $\widetilde{\Lambda}^2$ denotes the cutoff[4] for $\frac{1}{2}\widetilde{\not{D}^2}$, regularizing the divergence for small α. Now we can apply the heat kernel expansion

$$\mathrm{Tr}\, e^{-\alpha\widetilde{\not{D}^2}} = \sum_{n\geq 0} \alpha^{\frac{n-4}{2}} \int_{\mathcal{M}} d^4y\; a_n\left(y, \widetilde{\not{D}^2}\right) \tag{3.11}$$

where the Seeley-de Witt coefficients $a_n(y, \widetilde{\not{D}^2})$ are given by [15]

$$\begin{aligned} a_0(y) &= \frac{1}{16\pi^2}\,\mathrm{tr}\,\mathbb{1}\;, \\ a_2(y) &= \frac{1}{16\pi^2}\mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\;\mathbb{1}\;+\mathcal{E}\right) \\ \mathcal{E} &= -\widetilde{G}^{mn}\left(\partial_m\Omega_n + \Omega_m\Omega_n - \widetilde{\Gamma}^k_{mn}\Omega_k,\right) \qquad (3.12) \\ \Omega_m &= \frac{1}{2}\widetilde{G}_{mn}\left(e^{-\sigma}a^n + \widetilde{\Gamma}^n\right) \qquad (3.13) \end{aligned}$$

and tr denotes the trace over the spinorial matrices. The effective action is therefore

$$\Gamma_\Psi = \frac{1}{16\pi^2}\int d^4y\left(2\,\mathrm{tr}(\mathbb{1}\;)\,\widetilde{\Lambda}^4 + \mathrm{tr}\left(\frac{R[\tilde{G}]}{6}\;\mathbb{1}\;+\mathcal{E}\right)\widetilde{\Lambda}^2 + O(\log\tilde{\Lambda})\right)\;, \tag{3.14}$$

where $\mathrm{tr}(\mathbb{1}\;) = 4$ for a Dirac fermion. Everything is expressed in terms of the unimodular metric $\tilde{G}_{ab}$, which can be written in terms of G_{ab} using

$$\begin{aligned} R[\widetilde{G}] &= \rho(y)\left(R[G] + 3\Delta_G\sigma - \frac{3}{2}G^{ab}\partial_a\sigma\partial_b\sigma\right) \\ \Delta_G\sigma &= -(G^{ab}\partial_a\partial_b\sigma - \Gamma^c\partial_c\sigma) \\ \Gamma^a &= G^{bc}\Gamma^a_{bc} \\ e^{-\sigma(y)} &= \rho(y) = (\det G_{ab})^{1/4} \\ \widetilde{\Gamma}^a &= \widetilde{G}^{cd}\widetilde{\Gamma}^a_{cd} \;=\; e^{-\sigma}\Gamma^a - e^{-\sigma}\left(\partial_b\sigma\right)G^{ba}. \qquad (3.15) \end{aligned}$$

The actual computation of $\mathrm{tr}\,\mathcal{E}$ is given in section 4.2. Note the relative minus sign of the various terms in the effective action Γ_Ψ compared with the induced action due to a scalar field [4],

$$\Gamma_\Phi = \frac{1}{16\,\pi^2}\int d^4y\left(-2\widetilde{\Lambda}^4 - \frac{1}{6}R[\widetilde{G}]\widetilde{\Lambda}^2 + O(\log\widetilde{\Lambda})\right). \tag{3.16}$$

hence

$$\Gamma_\Psi + 4\,\Gamma_\Phi \;=\; \frac{1}{16\,\pi^2}\int d^4y\,\mathrm{tr}\,\mathcal{E}\,\widetilde{\Lambda}^2\,. \tag{3.17}$$

This expresses the cancellation of the induced actions due to fermions and bosons, apart from the $\mathcal{E}$ term. For the standard coupling of Dirac fermions to gravity on commutative spaces, one has [16]

$$\mathrm{tr}\,\mathcal{E}_{\mathrm{comm}} = -R \tag{3.18}$$

[4]We write $2\Lambda^2$ instead of Λ^2 in (3.10) in order to be consistent with the cutoff for $\frac{1}{2}\Delta^2$ for scalar fields used in [4], which is implicit in Γ_Φ as given below.

which originates from an additional constant term $-\frac{1}{4}R$ in $\not{D}^2_{\rm comm}$ (Lichnerowicz's formula). In our case, $\mathcal{E}$ turns out to be somewhat modified due to the vanishing spin connection, nevertheless it contains the appropriate curvature scalar plus an additional term (4.21). This will be discussed further in section 5.2.

4. Computation of the Seeley-DeWitt coefficient

In this section we determine the second Seeley-de Witt coefficient for (3.9). We first obtain an exact, compact result for the Ricci scalar $R[\widetilde{G}]$ expressed in terms of the Poisson tensor $\theta^{mn}(y)$. This may be useful for other purposes as well. We then compute $\mathrm{tr}\,\mathcal{E}$ in terms of $\theta^{mn}(y)$, which turns out to be closely related to $R[\widetilde{G}]$ as desired.

4.1 Ricci scalar in terms of θ^{mn}

The curvature is given as usual by

$$R_{abc}{}^{d} = \partial_b\Gamma^d_{ac} - \partial_a\Gamma^d_{bc} + \Gamma^e_{ac}\Gamma^d_{eb} - \Gamma^e_{bc}\Gamma^d_{ea}\,. \tag{4.1}$$

The Ricci scalar is then

$$R = G^{ac}\,R^b_{abc} = G^{ac}\left(\partial_b\Gamma^b_{ac} - \partial_a\Gamma^b_{bc} + \Gamma^e_{ac}\Gamma^b_{eb} - \Gamma^e_{bc}\Gamma^b_{ea}\right). \tag{4.2}$$

In terms of the metric and its derivatives R is given by

$$\begin{aligned} R \;=\; & \left(\partial_b G^{bd}\right) G^{ca}\left(\partial_a G_{cd}\right) + G^{ab}G^{cd}\partial_b\partial_d G_{ac} - G^{mn}G^{pq}\partial_p\partial_q G_{mn} \\ & -\left(\partial_b G^{bd}\right) G^{mn}\left(\partial_d G_{mn}\right) - \frac{3}{4}G^{pq}\left(\partial_p G^{mn}\right)\left(\partial_q G_{mn}\right) \\ & +\frac{1}{2}G^{np}\left(\partial_p G^{ac}\right)\left(\partial_c G_{na}\right) - \frac{1}{4}G^{pq}G^{mn}\left(\partial_p G_{mn}\right)G^{kl}\left(\partial_q G_{kl}\right). \end{aligned} \tag{4.3}$$

We use the explicit formula for the metric tensor

$$G^{mn}(y) = \theta^{ma}(y)\theta^{nb}(y)g_{ab} \tag{4.4}$$

to express R in terms of θ (see appendix A for details)

$$\begin{aligned} R \;=\; & -2\left(\partial_a\theta^{ap}\right)G^{bc}\left(\partial_c\theta^{-1}_{bp}\right) - \left(\partial_a\theta^{ap}\right)\left(\partial_b\theta^{bq}\right)g_{pq} \\ & +2G^{mp}G^{nq}\theta^{-1}_{ma}\partial_p\partial_q\theta^{-1}_{nb}\,g^{ab} + \frac{1}{2}G^{bc}\left(\partial_c\theta^{-1}_{ap}\right)G^{ad}\left(\partial_d\theta^{-1}_{bq}\right)g^{pq} \\ & +2\theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn} - \frac{1}{2}G^{mn}G^{pq}\left(\partial_p\theta^{-1}_{ma}\right)\left(\partial_q\theta^{-1}_{nb}\right)g^{ab} \\ & +4\theta^{bc}\left(\partial_b\theta^{da}\right)\left(\partial_d\sigma\right)g_{ca} + \frac{3}{2}G^{pq}\left(\partial_p\theta^{mn}\right)\left(\partial_q\theta^{-1}_{mn}\right) \\ & -G^{mp}\left(\partial_p\theta^{nq}\right)\left(\partial_n\theta^{-1}_{mq}\right) - \frac{1}{2}\left(\partial_n\theta^{cq}\right)\left(\partial_c\theta^{np}\right)g_{pq}. \end{aligned} \tag{4.5}$$

This equation holds in fact for any vielbein using the identification

$$\theta^{\mu m}g_{mb} = e^{\mu}{}_{b} \qquad \theta^{-1}_{\mu m}g^{mb} = -e_{\mu}{}^{b} \tag{4.6}$$

since we have not exploited any property of θ^{mn} yet. However, by making use of the Jacobi identity,

$$\partial_a \theta^{-1}_{bc} + \partial_c \theta^{-1}_{ab} + \partial_b \theta^{-1}_{ca} = 0 \tag{4.7}$$

$$\partial_a \theta^{pq} = -\left(\partial_c \theta^{-1}_{am}\right)\left(\theta^{mp}\theta^{cq} - \theta^{mq}\theta^{cp}\right), \tag{4.8}$$

several terms appearing in the computation of $\operatorname{tr}\mathcal{E}$ and $R[\widetilde{G}]$ are equivalent:[5]

$$\begin{aligned}
G^{mk}\left(\partial_k\theta^{na}\right)\left(\partial_n\theta^{-1}_{ma}\right) &= \frac{1}{2}G^{pq}\left(\partial_p\theta^{mn}\right)\left(\partial_q\theta^{-1}_{mn}\right)\\
\left(\theta^{mn}\partial_q\theta^{-1}_{mn}\right)\theta^{qa}G^{pk}\left(\partial_k\theta^{-1}_{pa}\right) &= 2\left(\partial_m\theta^{ma}\right)G^{nk}\left(\partial_k\theta^{-1}_{na}\right)\\
\left(\theta^{mn}\partial_q\theta^{-1}_{mn}\right)\theta^{qa}\left(\partial_p\theta^{pb}\right)g_{ab} &= 2\left(\partial_m\theta^{ma}\right)\left(\partial_n\theta^{nb}\right)g_{ab}\\
\theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn} &= -2G^{mp}G^{nq}\theta^{-1}_{ma}\partial_p\partial_q\theta^{-1}_{nb}g^{ab}\\
\theta^{ma}\partial_m\partial_n\theta^{nb}g_{ab} &= G^{mn}\partial_m\partial_n\sigma + \theta^{ma}\left(\partial_m\theta^{nb}\right)\left(\partial_n\sigma\right)g_{ab}\\
&= \frac{1}{2}\theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn} + \frac{1}{2}G^{pq}\left(\partial_p\theta^{mn}\right)\left(\partial_q\theta^{-1}_{mn}\right)\\
&\quad + \left(\partial_p\theta^{pa}\right)G^{qk}\left(\partial_k\theta^{-1}_{qa}\right)
\end{aligned} \tag{4.9}$$

Thus in our framework the Ricci scalar associated to G_{ab} takes the form

$$\begin{aligned}
R[G] = {}& \theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn} + G^{pq}\left(\partial_p\theta^{mn}\right)\left(\partial_q\theta^{-1}_{mn}\right)\\
&+2\left(\partial_m\theta^{ma}\right)G^{nk}\left(\partial_k\theta^{-1}_{na}\right) - G^{mn}\left(\partial_m\sigma\right)\left(\partial_n\sigma\right)\\
&+\frac{1}{2}G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab} - \frac{1}{2}G^{mn}G^{pq}\left(\partial_p\theta^{-1}_{ma}\right)\left(\partial_q\theta^{-1}_{nb}\right)g^{ab}\\
&-\frac{1}{2}\left(\partial_m\theta^{na}\right)\left(\partial_n\theta^{mb}\right)g_{ab}.
\end{aligned} \tag{4.10}$$

Evaluating also

$$\begin{aligned}
3\Delta_G\sigma - \frac{3}{2}G^{mn}\partial_m\sigma\partial_n\sigma = {}& -\frac{3}{2}G^{mn}\left(\partial_m\theta^{pq}\right)\left(\partial_n\theta^{-1}_{pq}\right) - \frac{3}{2}\theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn}\\
&+\frac{3}{2}G^{mn}\partial_m\sigma\partial_n\sigma - 3\left(\partial_m\theta^{ma}\right)G^{nk}\left(\partial_k\theta^{-1}_{na}\right)
\end{aligned} \tag{4.11}$$

gives

$$\begin{aligned}
R[\widetilde{G}] &= e^{-\sigma}\left(R\left[G\right] + 3\Delta_G\sigma - \frac{3}{2}G^{ab}\partial_a\sigma\partial_b\sigma_b\right)\\
&= e^{-\sigma}\Bigg[-\frac{1}{2}\theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn} - \frac{1}{2}G^{pq}\left(\partial_p\theta^{mn}\right)\left(\partial_q\theta^{-1}_{mn}\right)\\
&\quad -\left(\partial_m\theta^{ma}\right)G^{nk}\left(\partial_k\theta^{-1}_{na}\right) + \frac{1}{2}G^{mn}\left(\partial_m\sigma\right)\left(\partial_n\sigma\right)\\
&\quad +\frac{1}{2}G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab} - \frac{1}{2}G^{mn}G^{pq}\left(\partial_p\theta^{-1}_{ma}\right)\left(\partial_q\theta^{-1}_{nb}\right)g^{ab}\\
&\quad -\frac{1}{2}\left(\partial_m\theta^{na}\right)\left(\partial_n\theta^{mb}\right)g_{ab}\Bigg].
\end{aligned} \tag{4.12}$$

[5] By means of these relations one can also check that the action (3.7) is indeed hermitian.

Via partial integration, the number of independent terms can be further reduced:

$$\begin{aligned}
\int d^4y\, e^{-\sigma}\theta^{ma}\partial_m\partial_n\theta^{nb}g_{ab} &= 0,\\
\int d^4y\, e^{-\sigma}\left(\partial_m\theta^{na}\right)\left(\partial_n\theta^{mb}\right)g_{ab} &= -\int d^4y\,\frac{e^{-\sigma}}{2}\left(\theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn}+G^{pq}\left(\partial_p\theta^{mn}\right)\left(\partial_q\theta^{-1}_{mn}\right)\right),\\
\int d^4y\, e^{-\sigma}\left(\partial_p\theta^{pa}\right)G^{qk}\left(\partial_k\theta^{-1}_{qa}\right) &= \int d^4y\, e^{-\sigma}\left(\partial_m\theta^{na}\right)\left(\partial_n\theta^{mb}\right)g_{ab}\,. \qquad (4.13)
\end{aligned}$$

This yields the following compact form for the Ricci scalar

$$\begin{aligned}
\int d^4y\, R[\widetilde{G}]\widetilde{\Lambda}^2 = \int d^4y\, e^{-\sigma}\Big\{&\frac{1}{2}G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab}-\frac{1}{2}G^{mn}G^{pq}\left(\partial_p\theta^{-1}_{ma}\right)\left(\partial_q\theta^{-1}_{nb}\right)g^{ab}\\
&-\frac{1}{2}\left(\partial_p\theta^{pa}\right)G^{qk}\left(\partial_k\theta^{-1}_{qa}\right)+\frac{1}{2}G^{mn}\left(\partial_m\sigma\right)\left(\partial_n\sigma\right)\Big\}\widetilde{\Lambda}^2\,, \qquad (4.14)
\end{aligned}$$

which is indeed what we need because $\det\tilde{G}_{ab}=1$. However, one has to be careful when using partial integration. We regard here the cutoff $\widetilde{\Lambda}^2$ as a constant effective cutoff for $\Delta_{\widetilde{G}}$, independent of y^a. In section 5, we will relate $\widetilde{\Lambda}^2$ with the effective cutoff Λ for Δ_A, which is the Laplacian of NC gauge theory. This corresponds to a position-dependent $\widetilde{\Lambda}^2$ as given in (5.23). In that case, one must either include additional terms from partial integration, or use (4.12) as we will do.

4.2 Evaluation of $\mathrm{tr}\,\mathcal{E}$

We need to evaluate

$$\mathrm{tr}\mathcal{E} = -\mathrm{tr}\widetilde{G}^{ab}\left(\partial_a\Omega_b+\Omega_a\Omega_b-\widetilde{\Gamma}^r_{ab}\Omega_r\right), \qquad (4.15)$$

where

$$\begin{aligned}
\Omega_m &= \frac{1}{2}\widetilde{G}_{mn}\left(\widetilde{a}^n+\tilde{\Gamma}^n\right)\\
&= \frac{1}{2}\left(G_{mn}\gamma_a\gamma_b\theta^{pa}\left(\partial_p\theta^{nb}\right)-G_{mn}\left(\partial_pG^{pn}\right)+\partial_m\sigma\right) \qquad (4.16)
\end{aligned}$$

and $\tilde{a}^n = e^{-\sigma}a^n$. For the computation of $\mathrm{tr}\,\mathcal{E}$ we use the relations given in appendix A and the Jacobi identity (4.9). Recalling

$$\begin{aligned}
\mathrm{tr}\gamma^a\gamma^b &= 4g^{ab},\\
\mathrm{tr}\gamma^a\gamma^b\gamma^c\gamma^d &= 4\left(g^{ab}g^{cd}-g^{ac}g^{bd}+g^{ad}g^{bc}\right), \qquad (4.17)
\end{aligned}$$

we find for the individual parts of $\mathrm{tr}\,\mathcal{E}$:

$$\begin{aligned}
\mathrm{tr}\,\widetilde{G}^{mn}\partial_m\Omega_n = 2e^{-\sigma}\Big\{&G^{mn}\left(\partial_mG_{np}\right)g_{ab}\theta^{qa}\left(\partial_q\theta^{pb}\right)\\
&+\left(\partial_m\theta^{qa}\right)\left(\partial_q\theta^{mb}\right)g_{ab}+\theta^{ma}\partial_m\partial_n\theta^{nb}g_{ab}\\
&-G^{mn}\left(\partial_mG_{np}\right)\left(\partial_qG^{qp}\right)-\partial_m\partial_nG^{mn}+G^{mn}\partial_m\partial_n\sigma\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= 2e^{-\sigma}\Big\{ (\partial_m\theta^{ma})\, G^{nk}\left(\partial_k\theta^{-1}_{na}\right) - \theta^{ma}\partial_m\partial_n\theta^{nb}g_{ab} \\
&\qquad + \frac{1}{2}G^{pq}\,(\partial_p\theta^{mn})\left(\partial_q\theta^{-1}_{mn}\right) + \frac{1}{2}\theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn}\Big\} \\
&= 0
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
\mathrm{tr}\,\widetilde{G}^{mn}\Omega_m\Omega_n &= e^{-\sigma}\Big\{ (g_{ab}g_{cd} - g_{ac}g_{bd} + g_{ad}g_{bc})\,G_{nk}\theta^{qa}\left(\partial_q\theta^{nb}\right)\theta^{lc}\left(\partial_l\theta^{kd}\right) \\
&\qquad -2\,(\partial_q G^{qn})\,G_{nk}g_{cd}\theta^{lc}\left(\partial_l\theta^{kd}\right) + 2\theta^{qa}\left(\partial_q\theta^{nb}\right)(\partial_n\sigma)\,g_{ab} \\
&\qquad -2\left(\partial_l G^{lk}\right)(\partial_k\sigma) + (\partial_q G^{qn})\left(\partial_l G^{lk}\right)G_{nk} + G^{mn}\,(\partial_m\sigma)\,(\partial_n\sigma)\Big\} \\
&= e^{-\sigma}\Big\{ - G^{kl}G^{mn}\left(\partial_k\theta^{-1}_{ma}\right)\left(\partial_l\theta^{-1}_{nb}\right)g^{ab} + G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab}\Big\} \\
\mathrm{tr}\,\Omega_m\widetilde{\Gamma}^m &= e^{-\sigma}\mathrm{Tr}\,(\Omega_m\Gamma^m - \Omega_m G^m n\partial_n\sigma) \\
&= e^{-\sigma}\mathrm{Tr}\,(-\Omega_m\,(\partial_n G^{nm}) + \Omega_m G^{mn}\partial_n\sigma) \\
&= 0
\end{aligned}
\tag{4.19}
$$

Hence we obtain

$$
\mathrm{tr}\,\mathcal{E} = e^{-\sigma}\Big\{G^{kl}G^{mn}\left(\partial_k\theta^{-1}_{ma}\right)\left(\partial_l\theta^{-1}_{nb}\right)g^{ab} - G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab}\Big\}. \tag{4.20}
$$

Comparing with (4.14) for $\widetilde{\Lambda}^2$ regarded as constant cutoff of $\Delta_{\widetilde{G}}$, we can write this as

$$
\begin{aligned}
\int d^4y\;\mathrm{tr}\,\mathcal{E} &= \int d^4y\;\left(-2\,R[\tilde{G}] - (\partial_p\theta^{pa})\,G^{qk}\left(\partial_k\theta^{-1}_{qa}\right) + G^{mn}\,(\partial_m\sigma)\,(\partial_n\sigma)\right) \\
&\overset{\text{eom}}{=} \int d^4y\;\left(-2R[\tilde{G}] + G^{mn}\partial_m\sigma\partial_n\sigma\right),
\end{aligned}
\tag{4.21}
$$

assuming on-shell geometries (2.18) in the last line. This formula applies for Dirac fermions, and with an additional factor $\frac{1}{2}$ for Weyl fermions. It is remarkable that $\mathrm{tr}\,\mathcal{E}$ is essentially given by the appropriate curvature scalar $R[\tilde{G}]$, and a contribution from the dilaton-like scaling factor $\rho = e^{-\sigma}$. This is a very reasonable modification of the standard result (3.18), as desired.

5. Relation with gauge theory on $\mathbb{R}^4_\theta$

We now want to interpret the fermionic action (2.11) as action for a Dirac fermion on the Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$ coupled to a U(1) gauge field in the adjoint. This point of view is obtained by writing the general covariant coordinate resp. matrix Y^a as

$$
Y^a = X^a + \mathcal{A}^a\,. \tag{5.1}
$$

Here X^a are generators of the Moyal-Weyl quantum plane, which satisfy

$$
[X^a, X^b] = i\bar{\theta}^{ab}\,, \tag{5.2}
$$

where $\bar{\theta}^{ab}$ is a constant antisymmetric tensor. These are particular solutions of the equations of motion (2.17). The effective geometry (2.6) for the Moyal-Weyl plane is indeed flat, given by

$$\begin{aligned}
\bar{g}^{ab} &= \bar{\theta}^{ac}\,\bar{\theta}^{bd} g_{cd} \\
\tilde{g}^{ab} &= \bar{\rho}\,\bar{g}^{ab}, \qquad \det \tilde{g}^{ab} = 1 \\
\bar{\rho} &= (\det \bar{\theta}^{ab})^{-1/2} = |\bar{g}_{ab}|^{1/4} \equiv \Lambda_{\rm NC}^4 \,.
\end{aligned} \tag{5.3}$$

Consider now the change of variables

$$\mathcal{A}^a(x) = -\bar{\theta}^{ab} A_b(x) \tag{5.4}$$

where A_a are hermitian matrices, interpreted as smooth functions on $\mathbb{R}^4_{\bar{\theta}}$. Thus we can write

$$[Y^a, f] = [X^a + \mathcal{A}^a, f] = i\bar{\theta}^{ab}\left(\frac{\partial}{\partial x^b} f + i\,[A_b, f]\right) \equiv i\bar{\theta}^{ab} D_b f, \tag{5.5}$$

giving for the quadratic form (3.7)

$$\begin{aligned}
S_{\rm square} &= (2\pi)^2\, {\rm Tr}\; \Psi^\dagger \gamma_a\gamma_b \left[Y^a, \left[Y^b, \Psi\right]\right] \\
&= -\int d^4x\, \bar{\rho}\, \Psi^\dagger\; \gamma_a\gamma_b \bar{\theta}^{am}\bar{\theta}^{bn} D_m D_n \Psi \\
&= \int d^4x\, \Psi^\dagger\, \widetilde{\not{D}^2}_A \Psi \,.
\end{aligned} \tag{5.6}$$

This is an exact expression on $\mathbb{R}^4_\theta$, where

$$\widetilde{\not{D}^2}_A = -\bar{\rho}\, \gamma_a\gamma_b \bar{\theta}^{am}\bar{\theta}^{bn} D_m D_n \; = -\,\tilde{\gamma}^m \tilde{\gamma}^n\, D_m D_n \,, \tag{5.7}$$

and

$$\tilde{\gamma}^a = (\det \bar{g}_{ab})^{\frac{1}{8}}\, \gamma_b\, \bar{\theta}^{ba}, \qquad \{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\,\tilde{g}^{ab} \,. \tag{5.8}$$

We now want to rewrite the geometrical results of section 4 in terms of gauge theory on $\mathbb{R}^4_\theta$ in x-coordinates. To do this, note that most formulas of section 4 are not generally covariant, but only valid in the preferred y-coordinates defined by the matrix models where $g_{ab} = \delta_{ab}$ resp. $g_{ab} = \eta_{ab}$. (5.1) defines the leading-order relation between y and x coordinates,

$$y^a = x^a - \bar{\theta}^{ab}\bar{A}_b + O(\theta^2)\,, \tag{5.9}$$

with Jacobian

$$\begin{aligned}
\left|\frac{\partial y^a}{\partial x^b}\right| &= 1 - \bar{\theta}^{ac}\frac{\partial A_c}{\partial x^a} + O(\theta^2) \\
&= 1 - \frac{1}{2}\bar{\theta}^{mn}\bar{F}_{mn} + O(\theta^2).
\end{aligned} \tag{5.10}$$

See [4] for details of this change of variables. In order to avoid confusion we will denote all x-dependent tensors with a bar, and we write

$$\partial_a = \frac{\partial}{\partial y^a}, \qquad \bar{\partial}_a = \frac{\partial}{\partial x^a}. \tag{5.11}$$

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The Poisson tensor can be written in terms of the $\mathfrak{u}(1)$ field strength as

$$i\theta^{ab}(y) = \left[Y^a, Y^b\right] = i\bar{\theta}^{ab} - i\bar{\theta}^{ac}\bar{\theta}^{bd}\bar{F}_{cd}, \tag{5.12}$$

where $\bar{F}_{cd}$ is a rank two tensor in x coordinates on $\mathbb{R}^4_\theta$. This amounts to

$$\theta^{-1}_{ab} = \bar{\theta}^{-1}_{ab} - \bar{F}_{ab} \tag{5.13}$$

at leading order. We also need the metric (2.6) in x-coordinates,

$$\begin{aligned} G^{ab} &= \left(\bar{\theta}^{ac} - \bar{\theta}^{ai}\bar{\theta}^{cj}\bar{F}_{ij}\right)\left(\bar{\theta}^{bd} - \bar{\theta}^{be}\bar{\theta}^{df}\bar{F}_{ef}\right) g_{cd} \\ &= \bar{g}^{ac}\left(\delta^b_c + \bar{F}_{cd}\bar{\theta}^{db} + \bar{g}_{ce}\bar{\theta}^{ef}\bar{F}_{fd}\bar{g}^{db} + \bar{g}_{cd}\bar{\theta}^{de}\bar{F}_{ef}\bar{g}^{fg}\bar{F}_{gh}\bar{\theta}^{hb}\right) \\ &\equiv \bar{g}^{ac}\left(\delta^b_c + X^b_c\right). \end{aligned} \tag{5.14}$$

To compute the determinant, we use

$$\det\left(1 + X_{ij}\right) = 1 + trX + \frac{1}{2}\left((trX)^2 - tr\left(X^2\right)\right) + O(X^3). \tag{5.15}$$

Then

$$trX = -2\bar{F}_{mn}\bar{\theta}^{mn} - \bar{\theta}^{em}\bar{g}_{mn}\bar{\theta}^{fn}\bar{F}_{eh}\bar{g}^{hg}\bar{F}_{gf} \tag{5.16}$$

and

$$e^\sigma = |G^{ab}|^{1/4} = |\bar{g}^{ab}|^{1/4}\left(1 - \frac{1}{2}\bar{\theta}^{mn}\bar{F}_{mn} + O(\bar{\theta}^3)\right). \tag{5.17}$$

By a straightforward application of the relations given above one can now write down the second Seeley-de Witt coefficient in x-coordinates. $R[\tilde{G}]$ expressed in x-coordinates agrees with eq. (78) in [4], as it should be. $\operatorname{tr}\mathcal{E}$ in x-coordinates is given as

$$\begin{aligned} \operatorname{tr}\mathcal{E} &= |\bar{g}_{ab}|^{1/4}\left(-\frac{1}{4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} - \frac{1}{2}\bar{g}^{ac}\bar{g}^{bd}\bar{F}_{ab}\bar{\partial}^2\bar{F}_{cd}\right) \\ &\quad + \frac{1}{4}|\bar{g}_{ab}|^{1/4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \\ &= -|\bar{g}_{ab}|^{1/4}\frac{1}{2}\bar{g}^{ac}\bar{g}^{bd}\bar{F}_{ab}\bar{\partial}^2\bar{F}_{cd}, \end{aligned} \tag{5.18}$$

where

$$\bar{\partial}^2 = \bar{\partial}_a\bar{\partial}_b g^{ab}. \tag{5.19}$$

In appendix B this expansion is given explicitly for the individual terms. We have omitted $O(A)$ terms from both $R[\widetilde{G}]$ and $\operatorname{tr}\mathcal{E}$, which are total derivatives and do not contribute to the effective action. In this way, we find for the one-loop induced action

$$\begin{aligned} \Gamma_\Psi &= \int d^4y\left(a_0\widetilde{\Lambda}^4 + a_2\widetilde{\Lambda}^2 + O\left(\log\widetilde{\Lambda}\right)\right) \\ &= -4\Gamma_\Phi - \frac{1}{16\,\pi^2}\int d^4y\frac{\rho(y)}{2}\bar{g}^{ac}\bar{g}^{bd}\bar{F}_{ab}\bar{\partial}^2\bar{F}_{cd}\widetilde{\Lambda}^2. \end{aligned} \tag{5.20}$$

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Finally, there is a nontrivial relation between the cutoff $\widetilde{\Lambda}$ of the geometrical action and the cutoff Λ of the $\mathfrak{u}(1)$ gauge theory, which follows from the identity

$$S_{\rm square} = {\rm Tr}\, \Psi^\dagger \gamma_a \gamma_b \left[Y^a, \left[Y^b, \Psi\right]\right] = \int d^4y\, \Psi^\dagger \widetilde{\not{D}^2}_{\widetilde{G}} \Psi = \int d^4y \frac{\rho(y)}{\bar{\rho}} \Psi^\dagger \widetilde{\not{D}^2}_A \Psi. \quad (5.21)$$

For the Laplacians this means

$$\widetilde{\not{D}^2}_{\widetilde{G}} = \frac{\rho(y)}{\bar{\rho}} \widetilde{\not{D}^2}_A. \quad (5.22)$$

Since we implement the cutoffs using the Schwinger parameterization (3.10), they are related as follows (cf. [4])

$$\widetilde{\Lambda}^2 = \frac{\rho(y)}{\bar{\rho}} \Lambda^2. \quad (5.23)$$

This makes sense provided $\rho(y)/\bar{\rho}$ varies only on large scales respectively small momenta $p \ll \Lambda$, which is our working assumption. Together with (5.10), we obtain as a final result for the geometric one-loop effective action expressed in terms of gauge theory on $\mathbb{R}^4_\theta$

$$\begin{aligned} \Gamma_\Psi &= -4\Gamma_\Phi - \int d^4x \bar{\rho} \frac{\Lambda^2}{2} \bar{g}^{ac} \bar{g}^{bd} \bar{F}_{ab} \bar{\partial}^2 \bar{F}_{cd} \\ &= -4\Gamma_\Phi + \int \frac{d^4p}{(2\pi)^4} \widetilde{g}^{ac} \widetilde{g}^{bd} \bar{F}_{ab}(p) \bar{F}_{cd}(-p) \frac{p^2}{\Lambda_{\rm NC}^4} \frac{\Lambda^2}{2} \end{aligned} \quad (5.24)$$

where $p^2 = p_i p_j g^{ij}$. This agrees precisely with the one-loop computation in the gauge theory point of view obtained below. Note that the last term corresponds to ${\rm tr}\, \mathcal{E}$ in (3.17).

5.1 Comparison with UV/IR mixing

In this section, we compare the geometrical form of the one-loop effective action obtained in the previous section with the one-loop effective action obtained from the gauge theory point of view. The result is of course the same, which provides not only a nontrivial check for our geometrical interpretation, but also sheds new light on the conditions to which extent the semi-classical analysis of the previous section is valid. This generalizes the results of [4] to the fermionic case. We find as expected that the UV/IR mixing terms obtained by integrating out the fermions are given by the induced geometrical resp. gravitational action (3.14), in a suitable IR regime. In particular, we need an explicit, physical momentum cutoff Λ.

Using the variables and conventions of the previous section, the action (2.11) can be exactly rewritten as U(1) gauge theory on $\mathbb{R}^4_\theta$, which in the Euclidean case takes the form

$$\begin{aligned} S[\Psi] &= (2\pi)^2\, {\rm Tr} \Psi^\dagger \gamma_a [Y^a, \Psi] \\ &= \int d^4x\, \tilde{\Psi}^\dagger i \tilde{\gamma}^a (\bar{\partial}_a \tilde{\Psi} + ig[A_a, \tilde{\Psi}]) \end{aligned} \quad (5.25)$$

We introduce an explicit coupling constant g, and define a rescaled fermionic field

$$\tilde{\Psi} = |\bar{g}_{ab}|^{\frac{1}{16}}\, \Psi \quad (5.26)$$

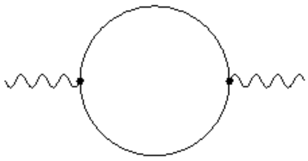

Figure 1: Fermionic one-loop diagram

in order to obtain the properly normalized effective metric $\tilde{g}^{ab}$; we will omit the tilde on Ψ henceforth. Recall also that only U(1) gauge fields are considered here, because only those correspond to the nontrivial geometry considered in the previous section.

We need the $O(A^2)$ contribution to the one-loop effective action obtained by integrating out the fermionic field Ψ. While this computation has been discussed several times in the literature [17, 18, 20−22], the known results are not accurate enough for our purpose, i.e. in the regime $p^2, \Lambda^2 < \Lambda_{\rm NC}^2$ where the semiclassical geometry is expected to make sense. We need to analyze carefully the IR regime of the well-known effective cutoff $\Lambda_{\rm eff}(p)$ (5.32) for non-planar graphs as $p \to 0$, keeping Λ fixed. In this regime the non-planar diagrams almost coincide with the planar diagrams, and the leading IR corrections due to the non-planar diagrams correspond to the induced geometrical terms in (3.14). This has not been considered in previous attempts to explain UV/IR mixing, e.g. in terms of exchange of closed string modes [23, 24].

To proceed one can either square the Dirac operator as in [21], or use directly the fermionic Feynman rules. We choose the latter approach here, and consider the Feynman diagram in figure 1 corresponding to

$$\begin{aligned}\Gamma_\Psi &= -\frac{1}{2}\mathrm{Tr}\log\Delta_0 - \frac{g^2}{2}\left\langle \int d^4x\, \rho\, \bar{\Psi}\tilde{\gamma}^a[A_a,\Psi] \int d^4y\, \rho\, \bar{\Psi}\tilde{\gamma}^b[A_b,\Psi]\right\rangle \\ &= -\frac{1}{2}\mathrm{Tr}\log\Delta_0 + \Gamma_\Psi(A). \end{aligned} \tag{5.27}$$

The minus sign in front is due to the fermionic loop. This gives

$$\begin{aligned}\Gamma_\Psi &= -4g^2 \int \frac{d^4p}{(2\pi)^4}\, A_{a'}(p)A_{b'}(-p)\,\tilde{g}^{a'a}\tilde{g}^{b'b} \int \frac{d^4k}{(2\pi)^4}\, \frac{2k_ak_b + k_ap_b + p_ak_b - \tilde{g}_{ab}k(k+p)}{(k\cdot k)((k+p)\cdot(k+p))} \\ &\quad \times\left(e^{-ik_i\theta^{ij}p_j} - 1\right)\end{aligned} \tag{5.28}$$

which is quite close to the bosonic case, using the notation

$$\begin{aligned} k\cdot k &\equiv k_ik_j\,\tilde{g}^{ij}\,. \\ k^2 &\equiv k_i\,k_jg^{ij}\,. \end{aligned} \tag{5.29}$$

The momentum integrals are understood to be regularized by a cutoff Λ, implemented via a Schwinger parameter as in [4]. To evaluate this loop integral, we rewrite it in a different way as in [21]

$$
\begin{aligned}
&-\int \frac{d^4k}{(2\pi)^4} \frac{4k_a k_b + 2k_a p_b + 2p_a k_b - 2\tilde{g}_{ab} k(k+p)}{(k\cdot k)((k+p)\cdot(k+p))} \big(e^{-ik_i\theta^{ij}p_j} - 1\big) \\
&\qquad = -\int \frac{d^4k}{(2\pi)^4} \Bigg(\frac{(2k_a + p_a)(2k_b + p_b) - (p_a p_b - \tilde{g}_{ab} p\cdot p)}{(k\cdot k)((k+p)\cdot(k+p))} \\
&\qquad\qquad - \tilde{g}_{ab}\left(\frac{1}{k\cdot k} + \frac{1}{(k+p)\cdot(k+p)}\right)\Bigg) \big(e^{-ik_i\theta^{ij}p_j} - 1\big) \\
&\qquad = -\int \frac{d^4k}{(2\pi)^4} \left(\frac{(2k_a + p_a)(2k_b + p_b)}{(k\cdot k)((k+p)\cdot(k+p))} - 2\frac{\tilde{g}_{ab}}{k\cdot k}\right) \big(e^{-ik_i\theta^{ij}p_j} - 1\big) \\
&\qquad\quad + (p_a p_b - \tilde{g}_{ab} p\cdot p) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k\cdot k)((k+p)\cdot(k+p))} \big(e^{-ik_i\theta^{ij}p_j} - 1\big) \qquad (5.30)
\end{aligned}
$$

where we replaced $\frac{1}{(k+p)\cdot(k+p)}$ by $\frac{1}{k\cdot k}$ under the integral (which does not make a difference in the regularization used here). Now the first term is precisely the induced action Γ_Φ obtained by integrating out a scalar field Φ [4], which is known to be gauge invariant. The second term is logarithmic and manifestly gauge-invariant. Therefore

$$
\begin{aligned}
\Gamma_\Psi &= -4\,\Gamma_\Phi + g^2 n_f \int \frac{d^4p}{(2\pi)^4} A_{a'}(p) A_{b'}(-p)\; \tilde{g}^{a'a} \tilde{g}^{b'b} \, (p_a p_b - \tilde{g}_{ab} p\cdot p) \\
&\quad \times \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k\cdot k)((k+p)\cdot(k+p))} \left(e^{-ik_i\theta^{ij}p_j} - 1\right) \\
&= -4\,\Gamma_\Phi - g^2 n_f \int \frac{d^4p}{(2\pi)^4} A_{a'}(p) A_{b'}(-p)\, \tilde{g}^{a'a} \tilde{g}^{b'b} \, (p_a p_b - \tilde{g}_{ab} p\cdot p) \\
&\quad \times \frac{1}{8\pi^2} \int_0^1 dz \left(K_0\left(2\sqrt{\frac{z(1-z)p\cdot p}{\Lambda^2}}\right) - K_0\left(2\sqrt{\frac{z(1-z)p\cdot p}{\Lambda_{\rm eff}^2}}\right)\right), \qquad (5.31)
\end{aligned}
$$

for Dirac fermions, where

$$
\Lambda_{\rm eff}^2 = \frac{1}{1/\Lambda^2 + \frac{1}{4}\frac{p^2}{\Lambda_{\rm NC}^4}} = \Lambda_{\rm eff}^2(p) \qquad (5.32)
$$

is the “effective” cutoff for non-planar graphs, and $\Lambda_{\rm NC}$ is defined in (5.3). For the standard evaluation of the $k-$ integration see e.g. [4]. To proceed we consider the IR regime

$$
\frac{p^2\Lambda^2}{\Lambda_{\rm NC}^4} < 1\,, \qquad (5.33)
$$

see figure 2. Then both Λ and $\Lambda_{\rm eff}$ are large, and we can use the asymptotic expansions

$$
K_0\left(2\sqrt{\frac{m^2}{\Lambda^2}}\right) = -\left(\gamma + \log\left(\sqrt{\frac{m^2}{\Lambda^2}}\right)\right) + O\left(\frac{m^2}{\Lambda^2}\log\left(\frac{\Lambda}{m}\right)\right) \qquad (5.34)
$$

which gives

$$
\begin{aligned}
\Gamma_\Psi + 4\,\Gamma_\Phi &\sim \frac{g^2 n_f}{16\pi^2} \int \frac{d^4p}{(2\pi)^4} A_{a'}(p) A_{b'}(-p)\, \tilde{g}^{a'a} \tilde{g}^{b'b}\, (p_a p_b - \tilde{g}_{ab} p\cdot p) \,\log\left(\frac{\Lambda_{\rm eff}^2}{\Lambda^2}\right) \\
&= -\frac{1}{2}\frac{g^2 n_f}{16\pi^2} \int \frac{d^4p}{(2\pi)^4} \bar{F}_{ab} \bar{F}_{a'b'} \tilde{g}^{a'a} \tilde{g}^{b'b} \,\log\left(\frac{\Lambda_{\rm eff}^2}{\Lambda^2}\right)\,. \qquad (5.35)
\end{aligned}
$$

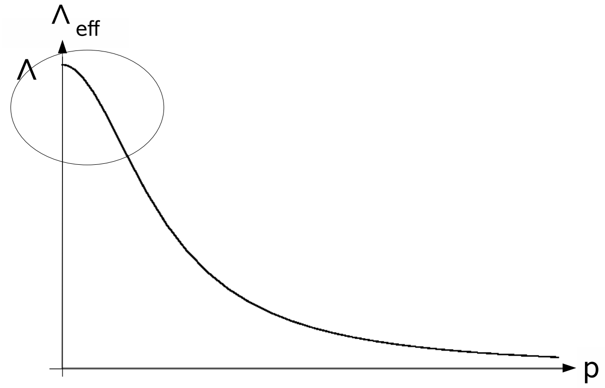

Figure 2: Relevant IR regime of $\Lambda_{\rm eff}(p)$

The only approximation here is the expansion (5.34) of the Bessel functions in (5.31). Γ_Φ is the 1-loop effective action for a (hermitian) scalar field as computed in [4],

$$\Gamma_\Phi = -\frac{g^2}{2}\frac{1}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\Bigg(-\frac{1}{6}\bar F_{ab}(p)\bar F_{a'b'}(-p)\tilde g^{a'a}\tilde g^{b'b}\log\left(\frac{\Lambda^2}{\Lambda^2_{\rm eff}}\right) \\ +\frac{1}{4}(\theta\bar F(p))(\theta\bar F(-p))\Big(\Lambda^4_{\rm eff}-\frac{1}{6}p\cdot p\,\Lambda^2_{\rm eff}+\frac{(p\cdot p)^2}{1800}\Big(47-30\log\Big(\frac{p\cdot p}{\Lambda^2_{\rm eff}}\Big)\Big)\Big)\Bigg). \quad (5.36)$$

These expressions are valid in the IR regime (5.33) $p\Lambda < \Lambda^2_{\rm NC}$ corresponding to "mild" UV/IR mixing. This is the same condition which was obtained for the bosonic case [4]. We can then use the expansions

$$\Lambda^2_{\rm eff} = \Lambda^2 - p^2\frac{\Lambda^4}{4\Lambda^4_{\rm NC}} + \cdots, \quad (5.37)$$

$$\Lambda^4_{\rm eff} = \Lambda^4 - p^2\frac{\Lambda^6}{2\Lambda^4_{\rm NC}} + \cdots, \quad (5.38)$$

$$\log\left(\frac{\Lambda^2}{\Lambda^2_{\rm eff}}\right) = \frac{1}{4}\frac{p^2\Lambda^2}{\Lambda^4_{\rm NC}} + \cdots \quad (5.39)$$

which gives

$$\Gamma_\Psi + 4\,\Gamma_\Phi \sim \frac{1}{4}\frac{g^2}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\,\tilde g^{a'a}\tilde g^{b'b}\,\bar F_{ab}(p)\bar F_{a'b'}(-p)\,\frac{p^2\Lambda^2}{\Lambda^4_{\rm NC}}, \\ = \frac{1}{4}\frac{g^2}{16\pi^2}\int\frac{d^4p}{(2\pi)^4}\,\bar\rho^2\Lambda^2p^2\,\bar g^{a'a}\bar g^{b'b}\,\bar F_{ab}(p)\bar F_{a'b'}(-p), \quad (5.40)$$

where $p^2 = p_ap_bg^{ab}$. There are obvious modifications due to the appropriate expansion of $\Lambda^2_{\rm eff}$ if one approaches the border of the IR regime (5.33).

To compare this with the geometrical results, we must take into account the different regularizations used in the heat-kernel expansion (3.10) and in the above one-loop computation. It was shown in [4] that these regularizations agree if we replace Λ^2 with $2\Lambda^2$

in the one-loop computation above.[6] We then find complete agreement with the result (5.24) obtained using the geometrical point of view. Notice in particular that the induced gravitational action is nontrivial even in the case of e.g. $N = 1$ supersymmetry. This is now understood in terms of induced gravity, and full cancellation is obtained only in the case of $N = 4$ supersymmetry. This will be discussed below.

Finally, Γ_Ψ and Γ_Φ can be related directly to the geometrical induced action (3.14) in a more restricted IR regime, as in [4]. Assume first that

$\Lambda \ll \Lambda_{\rm NC}$.

Then the IR regime (5.33) amounts to

$$p < \Lambda_{\rm NC}, \tag{5.41}$$

which is very reasonable range of validity for the classical gravity action. In this case,

$$\Lambda^6 \frac{p^2}{\Lambda_{\rm NC}^4} = \frac{\Lambda^4}{\Lambda_{\rm NC}^4} \Lambda^2 p^2 \;\ll\; \Lambda^2 p^2 \;\sim\; \Lambda^2 p \cdot p \tag{5.42}$$

so that we can replace

$$\Lambda_{\rm eff}^4 - \frac{1}{6} p \cdot p \Lambda_{\rm eff}^2 \;\sim\; \Lambda^4 - \frac{1}{6} p \cdot p \Lambda^2 \,. \tag{5.43}$$

Then the leading contribution to Γ_Φ is

$$\begin{aligned} \Gamma_\Phi \sim & -\frac{g^2}{2} \frac{1}{16\pi^2} \int \frac{d^4p}{(2\pi)^4} \Big(\frac{\Lambda^4}{4} (\theta \bar F(p))(\theta \bar F(-p)) - \frac{\Lambda^2}{24} \bar F_{ab}(p)\, \bar F_{a'b'}(-p) \frac{p^2}{\Lambda_{\rm NC}^4} \tilde g^{a'a} \tilde g^{b'b} \\ & \qquad - \frac{\Lambda^2}{24} (\theta \bar F(p))(\theta \bar F(-p)) p \cdot p \;+\; O(\log(\Lambda)) \;+ \text{finite} \Big) \\ = & -\frac{g^2}{2} \frac{1}{16\pi^2} \int d^4x \Big(\frac{\Lambda^4}{4} (\theta \bar F)(\theta \bar F) - \bar\rho \frac{\Lambda^2}{24} \Big(p^2\, \bar F_{ab}\, \bar F_{a'b'} \bar g^{a'a} \bar g^{b'b} + (p_a p_b \bar g^{ab})(\theta \bar F)(\theta \bar F) \Big) \\ & \qquad + O(\log(\Lambda)) \;+ \text{finite} \Big), \end{aligned} \tag{5.44}$$

where again $p^2 = p_a p_b g^{ab}$. Taking into account again the appropriate replacement $\Lambda^2 \to 2\Lambda^2$ corresponding to the geometrical regularization in (3.10) and setting $n_f = 2$ for Dirac fermions, one finds as in [4] complete agreement between the above result for Γ_Ψ with the result (3.14) obtained from the geometrical point of view.

Assume finally that the condition $\Lambda \ll \Lambda_{\rm NC}$ is violated, while maintaining the IR regime (5.33). Then there are additional terms in the effective action Γ_Φ beyond (5.44). They correspond to noncommutative corrections beyond the semi-classical geometrical terms in (3.16); we refer to [4] for some explicit results.

Gauge fields resp. gravitons The complete one-loop effective action for the gravitational sector sector of the basic matrix model (2.1) contains contributions of scalar fields, fermions, and "gauge fields". The latter means trace- U(1) components resp. gravitons

[6] while this was strictly speaking established only for the bosonic case, the argument should extend to the fermionic case without difficulties.

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inherent in $\tilde{G}_{ab}$, as well as possible nonabelian SU(n) components [1]. Ignoring nonabelian fields for now, it remains to compute the contributions of the U(1) gauge fields in a loop with 2 external A- legs. This would lead to a similar contribution, denoted by

$$e^{-\Gamma_A} = \int_{\text{one-loop}} dA\, e^{-S}\,. \tag{5.45}$$

While Γ_A is essentially straightforward to compute following e.g. [21], we can short-cut this computation by taking advantage of the finiteness of the $N = 4$ supersymmetric extension of the model. Note that the presence of particular scalar interaction terms in the $N = 4$ model is irrelevant for (5.45) at one loop. This leads to the result (5.46) given below.

5.2 Cancellations and supersymmetry

It is very interesting to compare the fermionic and the bosonic contribution to the gravitational action. As is well-known [17, 21], we note that the fermionic contribution to the one-loop effective action in NC gauge theory does not quite cancel the scalar contribution, due to (5.35). This means that even in supersymmetric cases some UV/IR mixing may remain. From the geometrical point of view, this terms corresponds to a gravitational action $\mathrm{tr}\,\mathcal{E}\,\tilde\Lambda^2 = -2R[\tilde{G}]\,\tilde\Lambda^2 + \cdots$, so that the cutoff $\tilde\Lambda^2$ should be interpreted as effective gravitational constant $\frac{1}{G}$. This is completely analogous to the commutative case, where the gravitational term (3.18) is induced. The remaining UV/IR mixing term cancels only in the case of $N = 4$ supersymmetry. We can therefore identify $\tilde\Lambda$ as the scale of $N = 4$ SUSY breaking (assuming such a model), above which the model is finite. These observations strongly suggest that for the model to be well-defined at the quantum level, $N = 4$ SUSY is required above the gravity scale i.e. the Planck scale. This is realized by the IKKT model [5] on a NC background.

Let us therefore consider an extension of the matrix model (2.1) with n_S scalar fields (= hermitian matrices ϕ^i) and n_Ψ Dirac fermions (hence $n_\Psi = \frac{1}{2}$ for Weyl fermions). The model with $N = 4$ SUSY has $n_S = 6$ and $n_\Psi = 2$, in addition to the U(1) gauge field A_μ resp. the graviton h_{ab}. Taking its finiteness for granted, it follows that

$$\Gamma_A = -2\,\Gamma_\Psi - 6\,\Gamma_\Phi\,. \tag{5.46}$$

This also holds for the SU(n) components of the gauge fields in a nonabelian versions of this $N = 4$ model; cf. [1]. Using (3.17), this can be written explicitly as

$$\Gamma_A = \frac{1}{16\,\pi^2}\int d^4y\,\left(-4\widetilde\Lambda^4 - \left(\frac{1}{3}R[\widetilde{G}] + 2\mathrm{tr}\,\mathcal{E}\right)\widetilde\Lambda^2 + O\left(\log\widetilde\Lambda\right)\right)\,. \tag{5.47}$$

This almost coincides with the contribution of 2 scalars (3.16), except for $\mathrm{tr}\,\mathcal{E}$ which stands for (4.21). At this point, it is important to keep in mind that a term $\int d^4y\,\sqrt{|\tilde{G}|}\,\widetilde\Lambda^4$ in the framework of emergent gravity does *not* amount to a large cosmological constant, as discussed in [1, 4]; see also [25]. We only point out here that flat space is a solution even in the presence of this term. These issues will be discussed elsewhere in more detail.

$N = 4$ supersymmetry transformations. The explicit form of the $N = 4$ SUSY transformations can be obtained as follows. Consider the basic matrix model (2.1) in 10 dimensions with fermions, with action

$$S = -\mathrm{Tr}\left(\frac{1}{2}[Y^a, Y^b][Y^{a'}, Y^{b'}]\eta_{aa'}\eta_{bb'} + \bar{\Psi}\gamma_a[Y^a, \Psi]\right), \qquad a = 0, \ldots, 9, \tag{5.48}$$

where Ψ is a 10-dimensional Majorana-Weyl spinor. It is well known that this action enjoys the 2 "matrix" supersymmetries [5]

$$\begin{aligned} \delta^{(1)}\Psi &= \frac{i}{2}[Y^a, Y^b]\gamma_{ab}\epsilon, & \delta^{(1)}Y^a &= i\bar{\epsilon}\gamma^a\Psi, \\ \delta^{(2)}\Psi &= \xi, & \delta^{(2)}Y^a &= 0 \end{aligned} \tag{5.49}$$

where $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$, and ϵ, ξ are Grassmann-valued spinors. To recover spacetime supersymmetry, we split the matrices into 4 + 6 dimensions as

$$Y^a = (Y^\mu, \phi^i), \qquad \mu = 0, \ldots, 3, \;\; i = 1, \ldots, 6. \tag{5.50}$$

Then the 4-dimensional Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$ is a (BPS) solution of the generalized matrix equations of motion, embedded as

$$\begin{aligned} Y^\mu &= \bar{X}^\mu, \qquad \mu = 0, \ldots, 3, \\ \phi^i &= 0\,. \end{aligned} \tag{5.51}$$

All previous geometrical considerations can be generalized, except that the matrix model now contains scalar fields $\phi^i(y)$. Even though we could consider a general metric $\tilde{G}_{ab}$ as above, let us for simplicity only discuss the point of view of U(1) gauge theory on $\mathbb{R}^4_\theta$. If we set $\xi = \frac{1}{2}\bar{\theta}^{\mu\nu}\gamma_{\mu\nu}\epsilon$ following [5] and use $\gamma_{a(3+i)} = \gamma_a\gamma_{3+i}$ recalling $Y^\mu = \bar{X}^\mu - \bar{\theta}^{\mu\nu}A_\nu$ and (5.8), then the combined transformation $\delta = \delta^{(1)} + \delta^{(2)}$ takes the form

$$\begin{aligned} \delta\Psi &= -i\bar{\rho}^{-1}\, F_{\mu\nu}\tilde{\Sigma}^{\mu\nu}\epsilon - \frac{1}{\sqrt{\bar{\rho}}}\,\tilde{\gamma}^\mu\partial_\mu\phi^i\gamma_{3+i}\epsilon \\ \delta\phi^i &= i\bar{\epsilon}\gamma^{3+i}\Psi, \\ \delta A_\nu &= -i\sqrt{\bar{\rho}}\,\tilde{g}_{\nu\mu}\bar{\epsilon}\tilde{\gamma}^\mu\Psi \end{aligned} \tag{5.52}$$

where $\tilde{\Sigma}^{\mu\nu} = \frac{i}{4}\,[\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]$. The constant factors of $\bar{\rho}$ can be absorbed by rescaling the fields. Noting that $\gamma_{3+i} = \gamma_5\,\Delta_i$ where Δ_i define the 6-dimensional Euclidean Clifford algebra, this indeed amounts to (abelian) $N = 4$ supersymmetry on $\mathbb{R}^4_\theta$.

In the case of general NC backgrounds, the SUSY transformation will also act on the metric $\tilde{G}_{\mu\nu}(y)$. This can be viewed as a supersymmetric form of emergent gravity, which will be worked out elsewhere. It also means that nontrivial background geometries "spontaneously" break $N = 4$ SUSY, as desired.

6. Discussion and outlook

In this paper, fermions are studied in the framework of emergent noncommutative gravity, as realized through matrix models of Yang-Mills type. The matrix model strongly suggests a particular fermionic term in the action, corresponding to a specific coupling to a background geometry with nontrivial metric $\tilde{G}_{\mu\nu}$. This coupling is similar to the standard coupling of fermions to a gravitational background, except that the spin connection vanishes in the preferred frame defined by the matrix model.

The main result of this paper is that in spite of this unusual feature, the resulting fermionic action is very reasonable, and properly describes fermions coupled to emergent gravity. In the point particle limit, fermions propagate along the appropriate trajectories, albeit with a different rotation of the spin. At the quantum level, we find an induced gravitational action which includes the expected Einstein-Hilbert term with a modified coefficient, as well as an additional term for a scalar density reminiscent of a dilaton. There are further terms which vanish for on-shell geometries. We conclude that the framework of emergent gravity does extend to fermions in a reasonable manner, and might well provide - in a suitable extension - a physically viable theory of gravity.

In a second part of the paper, we compare this induced gravitational action with the well-known UV/IR mixing in NC gauge theory due to fermions. Generalizing the results in [4] for scalar fields, we find as expected that the UV/IR mixing can be explained precisely by the gravitational point of view. This also provides a nice understanding for the fact that some UV/IR mixing remains in supersymmetric cases, and only disappears for $N = 4$ supersymmetry. The reason is that a gravitational action is induced even in supersymmetric cases, except in $N = 4$ SUSY. This in turn leads to the conjecture that the gravitational constant should be related to the scale of $N = 4$ SUSY breaking, which is quite reasonable. All of these findings suggest that the IKKT model on a noncommutative background [5–8] should be the most promising candidate for a realistic version of emergent gravity. These issues will be discussed in more detail elsewhere.

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A. Computation of R

We quote some identities which appear in the computation of $R[G]$ and $R[\widetilde{G}]$ in terms of θ-vielbeins:

$$\begin{aligned} \left(\partial_b G^{bd}\right) G^{ca} \left(\partial_a G_{cd}\right) &= -G^{ac} \left(\partial_c \theta^{-1}_{ap}\right) G^{bd} \left(\partial_d \theta^{-1}_{bq}\right) g^{pq} - 2 \left(\partial_a \theta^{ap}\right) G^{bc} \left(\partial_c \theta^{-1}_{bp}\right) \\ &\quad - \left(\partial_a \theta^{ap}\right) \left(\partial_b \theta^{bq}\right) g_{pq} \end{aligned} \tag{A.1}$$

$$\begin{aligned} G^{ab} G^{cd} \partial_b \partial_d G_{ac} &= 2 G^{mp} G^{nq} \theta^{-1}_{ma} \partial_p \partial_q \theta^{-1}_{nb} g^{ab} + G^{bc} \left(\partial_c \theta^{-1}_{ap}\right) G^{ad} \left(\partial_d \theta^{-1}_{bq}\right) g^{pq} \\ &\quad + G^{ad} \left(\partial_d \theta^{-1}_{ap}\right) G^{bc} \left(\partial_c \theta^{-1}_{bq}\right) g^{pq} \end{aligned} \tag{A.2}$$

$$G^{mn} G^{pq} \partial_p \partial_q G_{mn} = -2 \theta^{mn} G^{pq} \partial_p \partial_q \theta^{-1}_{mn} + 2 G^{mn} G^{pq} \left(\partial_p \theta^{-1}_{ma}\right) \left(\partial_q \theta^{-1}_{nb}\right) g^{ab} \tag{A.3}$$

$$\left(\partial_b G^{bd}\right) \left(\partial_d \sigma\right) = G^{mn} \left(\partial_m \sigma\right) \left(\partial_n \sigma\right) + \theta^{bc} \left(\partial_b \theta^{da}\right) \left(\partial_d \sigma\right) g_{ca} \tag{A.4}$$

$$
\begin{aligned}
G^{pq}\left(\partial_p G^{mn}\right)\left(\partial_q G_{mn}\right) &= -2G^{pq}\left(\partial_p \theta^{mn}\right)\left(\partial_q \theta^{-1}_{mn}\right) \\
&\quad -2G^{mn}G^{pq}\left(\partial_p \theta^{-1}_{ma}\right)\left(\partial_q \theta^{-1}_{nb}\right) g^{ab} \qquad \text{(A.5)} \\
G^{np}\left(\partial_p G^{ac}\right)\left(\partial_c G_{na}\right) &= -2G^{mp}\left(\partial_p \theta^{nq}\right)\left(\partial_n \theta^{-1}_{mq}\right) - \left(\partial_n \theta^{cq}\right)\left(\partial_c \theta^{np}\right) g_{pq} \\
&\quad -G^{nk}\left(\partial_k \theta^{-1}_{cp}\right) G^{cl}\left(\partial_l \theta^{-1}_{nq}\right) g^{pq} \\
G^{mn}\partial_m \sigma \partial_n \sigma &= \frac{1}{4} G^{pq}\theta^{mn}\left(\partial_q \theta^{-1}_{mn}\right)\theta^{kl}\left(\partial_q \theta^{-1} kl\right) \\
G^{mn}\partial_m \partial_n \sigma &= \frac{1}{2} G^{pq}\left(\partial_p \theta^{mn}\right)\left(\partial_q \theta^{-1}_{mn}\right) + \frac{1}{2}\theta^{mn} G^{pq}\partial_p \partial_q \theta^{-1}_{mn} \qquad \text{(A.6)}
\end{aligned}
$$

Below are some identities which have not appeared in the Ricci scalar but appear in the computation of $\mathrm{tr}\mathcal{E}$:

$$
\begin{aligned}
G^{mn}\left(\partial_m G_{np}\right) g_{ab}\theta^{qa}\left(\partial_q \theta^{pb}\right) &= -G^{mk}\left(\partial_k \theta^{-1}_{ma}\right) G^{nl}\left(\partial_l \theta^{-1}_{nb}\right) g^{ab} - \left(\partial_m \theta^{ma}\right) G^{nk}\left(\partial_k \theta^{-1}_{na}\right) \\
\partial_m \partial_n G^{mn} &= 2\theta^{ma}\partial_m \partial_n \theta^{nb} g_{ab} + \left(\partial_m \theta^{na}\right)\left(\partial_n \theta^{mb}\right) g_{ab} \\
&\quad + \left(\partial_m \theta^{ma}\right)\left(\partial_n \theta^{nb}\right) g_{ab} \\
G_{nk}\theta^{qa}\left(\partial_q \theta^{nb}\right)\theta^{lc}\left(\partial_l \theta^{kd}\right) g_{ab} g_{cd} &= G^{mk}\left(\partial_k \theta^{-1}_{ma}\right) G^{nl}\left(\partial_l \theta^{-1}_{nb}\right) g^{ab} \\
G_{nk}\theta^{qa}\left(\partial_q \theta^{nb}\right)\theta^{lc}\left(\partial_l \theta^{kd}\right) g_{ac} g_{bd} &= G^{kl} G^{mn}\left(\partial_k \theta^{-1}_{ma}\right)\left(\partial_l \theta^{-1}_{nb}\right) g^{ab} \\
G_{nk}\theta^{qa}\left(\partial_q \theta^{nb}\right)\theta^{lc}\left(\partial_l \theta^{kd}\right) g_{ad} g_{bc} &= G^{mp}\left(\partial_p \theta^{-1}_{na}\right) G^{nq}\left(\partial_q \theta^{-1}_{mb}\right) g^{ab} \\
G_{mn}\left(\partial_p G^{pm}\right)\left(\partial_q G^{qn}\right) &= \left(\partial_p \theta^{pa}\right)\left(\partial_q \theta^{qb}\right) g_{ab} + 2\left(\partial_p \theta^{pa}\right) G^{qk}\left(\partial_k \theta^{-1}_{qa}\right) \\
&\quad + G^{mk}\left(\partial_k \theta^{-1}_{ma}\right) G^{nl}\left(\partial_l \theta^{-1}_{nb}\right) g^{ab} \qquad \text{(A.7)}
\end{aligned}
$$

B. Expressing R and $\mathrm{tr}\,\mathcal{E}$ in x coordinates

Let us rewrite the terms which compose R and $\mathrm{tr}\mathcal{E}$ in terms of the $\mathfrak{u}(1)$ gauge fields. We will need [4]

$$
\begin{aligned}
e^{\sigma} &= \left(\det G^{ab}\right)^{1/4} = \left(\det \bar{g}^{ab}\right)^{1/4}\left(1 - \frac{1}{2}\bar{\theta}^{mn}\bar{F}_{mn} + O(\bar{\theta}^2)\right), \\
G^{ab} &= \left(\bar{\theta}^{ac} - \bar{\theta}^{ai}\bar{\theta}^{cj}\bar{F}_{ij}\right)\left(\bar{\theta}^{bd} - \bar{\theta}^{be}\bar{\theta}^{df}\bar{F}_{ef}\right) g_{cd}, \\
\theta^{ab} &= \bar{\theta}^{ab} - \bar{\theta}^{ac}\bar{\theta}^{bd}\bar{F}_{cd}, \\
\theta^{-1}_{ab} &= \bar{\theta}^{-1}_{ab} - \bar{F}_{ab}. \qquad \text{(B.1)}
\end{aligned}
$$

and denote $|\det \bar{g}_{ab}| \equiv |\bar{g}_{ab}|$. For the following terms this gives to $O(A^2)$ in x-coordinates

$$
\begin{aligned}
\int d^4y\, e^{-\sigma}\, G^{mk}\left(\partial_k \theta^{-1}_{ma}\right) G^{nl}\left(\partial_l \theta^{-1}_{nb}\right) g^{ab} &= -\frac{1}{4}\int d^4x\, |\bar{g}_{ab}|^{1/4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \\
\int d^4y\, e^{-\sigma}\left(\partial_m \theta^{ma}\right) G^{nk}\left(\partial_k \theta^{-1}_{na}\right) &= -\frac{1}{4}\int d^4x\, |\bar{g}_{ab}|^{1/4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \\
\int d^4y\, e^{-\sigma}\left(\partial_m \theta^{na}\right)\left(\partial_n \theta^{mb}\right) g_{ab} &= -\frac{1}{4}\int d^4x\, |\bar{g}_{ab}|^{1/4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq}
\end{aligned}
$$

$$
\begin{aligned}
\int d^4y\, e^{-\sigma}\, \theta^{ma}\partial_m\partial_n\theta^{nb}g_{ab} &= -\int d^4x\, |\bar{g}_{ab}|^{1/4}\Big(\frac{1}{2}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \\
&\qquad +\frac{1}{4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq}\Big) \\
\int d^4y\, e^{-\sigma}\, G^{pq}\left(\partial_p\theta^{mn}\right)\left(\partial_q\theta^{-1}_{mn}\right) &= -\frac{1}{2}\int d^4x\, |\bar{g}_{ab}|^{1/4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \\
\int d^4y\, e^{-\sigma}\, \theta^{mn}G^{pq}\partial_p\partial_q\theta^{-1}_{mn} &= \int d^4x\, |\bar{g}_{ab}|^{1/4}\Big(-\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \\
&\qquad +\frac{1}{2}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq}\Big) \\
\int d^4y\, e^{-\sigma}\, G^{kl}G^{mn}\left(\partial_k\theta^{-1}_{ma}\right)\left(\partial_l\theta^{-1}_{nb}\right)g^{ab} &= \int d^4x\, |\bar{g}_{ab}|^{1/4}\Big(-\frac{1}{4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \\
&\qquad -\frac{1}{2}\bar{g}^{ac}\bar{g}^{bd}\bar{F}_{ab}\bar{\partial}^2\bar{F}_{cd}\Big) \\
\int d^4y\, e^{-\sigma}\, G^{mk}\left(\partial_k\theta^{-1}_{na}\right)G^{nl}\left(\partial_l\theta^{-1}_{mb}\right)g^{ab} &= -\frac{1}{4}\int d^4x\, |\bar{g}_{ab}|^{1/4}\bar{\theta}^{mn}\bar{F}_{mn}\bar{\partial}^a\bar{\partial}_a\bar{\theta}^{pq}\bar{F}_{pq} \qquad \text{(B.2)}
\end{aligned}
$$

where we used [4]

$$
\begin{aligned}
\int d^4x\, \bar{F}_{mn}\bar{\theta}^{mn}\bar{\partial}^a\bar{\partial}_a\bar{F}_{pq}\bar{\theta}^{pq} &= 4\int d^4x\, g^{fh}\bar{F}_{fa}\bar{\partial}^a\bar{\partial}^s\bar{F}_{hs}, \\
\int d^4x\, g^{ab}\bar{F}_{am}\bar{\partial}^m\bar{\partial}^s\bar{F}_{bs} &= \int d^4x\, \frac{1}{2}\bar{g}^{ah}\bar{g}^{mr}\bar{F}_{hm}\bar{\partial}^2\bar{F}_{ra} - g^{si}\bar{g}^{ah}\bar{F}_{hs}\bar{\partial}^l\bar{\partial}_l\bar{F}_{ia}. \qquad \text{(B.3)}
\end{aligned}
$$

C. a^k and the spin connection

We observe that using the Jacobi equation, the linear term a^k appearing in $\not{D}^2$ (3.5) can be written in the form

$$
\begin{aligned}
a^k &= G^{kl}\partial_l\sigma + \frac{1}{4}\left[\gamma_a,\gamma_b\right]\theta^{lk}\left(\partial_l\theta^{ab}\right) - \Gamma^k \\
&= \frac{1}{4}\left[\gamma_a,\gamma_b\right]\theta^{lk}\left(\partial_l\theta^{ab}\right) + \theta^{ka}G^{mn}\left(\partial_n\theta^{-1}_{ma}\right) \\
&\overset{\text{eom}}{=} \frac{1}{4}\left[\gamma_a,\gamma_b\right]\theta^{lk}\left(\partial_l\theta^{ab}\right), \qquad \text{(C.1)}
\end{aligned}
$$

where the last line holds only for on-shell geometries, using the equations of motion. In the standard form of the Dirac operator, this linear term would have the form

$$
\begin{aligned}
a^k_{\text{comm}} &= \frac{1}{4}\left[\gamma_a,\gamma_b\right]\omega^{a\ b}_{\ l}G^{lk} - \Gamma^k \\
&= \frac{1}{4}\left[\gamma_a,\gamma_b\right]\left(\theta^{kd}g_{dm}\left(\partial_c\theta^{mb}\right)g^{ca} + \frac{1}{2}\theta^{lk}\left(\partial_l\theta^{ab}\right)\right) - G^{kl}\partial_l\sigma, \qquad \text{(C.2)}
\end{aligned}
$$

where

$$
\begin{aligned}
\omega^{a\ b}_{\ k} &= -\theta^{-1}_{lc}g^{ca}\nabla_k\theta^{lb} \\
&= -\theta^{-1}_{lc}g^{ca}\partial_k\theta^{lb} - \theta^{-1}_{lc}g^{ca}\Gamma^l_{km}\theta^{mb}
\end{aligned}
$$

$$= \frac{1}{2}\Big(\theta^{lb}\left(\partial_k\theta^{-1}_{lc}\right)g^{ca} - \theta^{na}\left(\partial_k\theta^{-1}_{nc}\right)g^{cb}$$
$$-\theta^{mb}\left(\partial_m\theta^{-1}_{kc}\right)g^{ca} + \theta^{ma}\left(\partial_m\theta^{-1}_{kc}\right)g^{cb}$$
$$-\theta^{mb}\left(\partial_m\theta^{na}\right)G_{nk} + \theta^{ma}\left(\partial_m\theta^{nb}\right)G_{nk}\Big) = -\omega^b{}_k{}^a \tag{C.3}$$

is the spin connection, using the explicit form of the frame (2.16). Note that the effect of the spin connection is different in our case. Nevertheless, it turns out that $\mathrm{tr}\,\mathcal{E}$ can be rewritten as Ricci scalar $R[\tilde{G}]$ plus a term dependent on σ, as shown in section 4.2.

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